

MSC 11L05, 11N37

Short Kloosterman sums to powerful modulus

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Abstract. We obtain the estimate of incomplete Kloosterman sum to powerful modulus q . The length N of the sum lies in the interval $e^{c(\log q)^{2/3}} \leq N \leq \sqrt{q}$.

Keywords: Kloosterman sums, powerful moduli, method of Postnikov

The aim of this paper is to estimate a short Kloosterman sum

$$S = S_q(N; a, b, c) = \sum'_{c < n \leq c+N} e_q(an^* + bn), \quad (1)$$

to powerful modulus q . Here N, a, b, c are integers, $1 < N \leq \sqrt{q}$, $(a, q) = 1$, the prime sign in the sum means the summation over $(n, q) = 1$, $nn^* \equiv 1 \pmod{q}$ and $e_q(v) = e^{2\pi i v/q}$. The number q is called powerful if its kernel $d = \prod_{p|q} p$ is small relative to q in the logarithmic scale. The simplest case of such numbers are $q = p^n$ where p is fixed prime and $n \rightarrow +\infty$.

For many times, A.A. Karatsuba [1]-[3] pointed out to the possibility of estimating of such sums with $q = p^n$ by method of A.G. Postnikov [4], [5]. In this paper, we prove the following statement.

THEOREM 1. *Suppose that $q \geq q_0$ is sufficiently large, $d = \prod_{p|q} p$ is the kernel of q , $\gamma_1 = 900$, $\gamma = 160^{-4}$ and let $\max(d^{15}, e^{\gamma_1(\ln q)^{2/3}}) \leq N \leq \sqrt{q}$. Then, for any a, b, c such that $(a, q) = 1$, the following estimate holds:*

$$|S_q(N; a, b, c)| < N \exp\left(-\gamma \frac{(\ln N)^3}{(\ln q)^2}\right). \quad (2)$$

The proof does not use any new ideas and is based only on the technic of papers [6], [7] and [8].

We need the following auxilliary assertions.

LEMMA 1. *Suppose that $0 < \varepsilon < 1$, $m = [2\varepsilon^{-1}]$, $q = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $d = \prod_{p|q} p$, and let $q_\varepsilon = dp_1^{\beta_1} \dots p_s^{\beta_s}$, where $\beta_r = [\varepsilon \alpha_r]$. Then for any z we have*

$$(1 + zq_\varepsilon)^* \equiv 1 - zq_\varepsilon + (zq_\varepsilon)^2 - \dots + (-1)^m (zq_\varepsilon)^m \pmod{q}. \quad (3)$$

REMARK. This assertion is an analogue of Postnikov's formula for $\text{ind}(1 + pz)$ modulo $q = p^n$, $p \geq 3$ (see [4], [5]). The idea of introducing the factor q_ε belongs to H. Iwaniec [8].

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PROOF. It is sufficient to prove that q_ε^{m+1} is divisible by q . This fact follows from easy-to-check inequalities $(m+1)(\beta_r+1) \geq \alpha_r$, $r = 1, 2, \dots, s$.

LEMMA 2. *Suppose that $P \geq 1$ and α are any real numbers. Then*

$$\left| \sum_{1 \leq n \leq P} e(\alpha n) \right| \leq \min(P, \|\alpha\|^{-1}),$$

where $e(z) = e^{2\pi iz}$ and $\|\alpha\| = \min(\{\alpha\}, 1 - \{\alpha\})$.

LEMMA 3. *Suppose that*

$$\alpha = \frac{A}{Q} + \frac{\theta}{Q^2}, \quad (A, Q) = 1, \quad Q \geq 1, \quad |\theta| \leq 1.$$

and let β , $U > 0$, $P \geq 1$ be any real numbers. Then the following inequality holds:

$$\sum_{1 \leq n \leq P} \min(U, \|\alpha n + \beta\|^{-1}) \leq 6 \left(\frac{P}{Q} + 1 \right) (U + Q \log Q).$$

For the proofs of lemmas 2,3, see, for example, [9, Ch. VI, §2].

Denote by $J_{k,m}(P; \lambda_1, \dots, \lambda_m)$ the number of solutions of the following system:

$$\begin{cases} x_1 + \dots + x_k = x_{k+1} + \dots + x_{2k} + \lambda_1 \\ \dots \\ x_1^m + \dots + x_k^m = x_{k+1}^m + \dots + x_{2k}^m + \lambda_m \end{cases}$$

with integers variables $1 \leq x_1, \dots, x_{2k} \leq P$ and let $J_{k,m}(P) = J_{k,m}(P; 0, \dots, 0)$. Obviously, $J_{k,m}(P; \lambda_1, \dots, \lambda_m) \leq J_{k,m}(P)$ for any $\lambda_1, \dots, \lambda_m$.

LEMMA 4 (VINOGRADOV'S MEAN VALUE THEOREM). *Suppose that $\tau \geq 1$, $k \geq m\tau$, $P \geq 1$ are integers. Then $J_{k,m}(P) \leq D(m, \tau) P^{2k - \Delta(m, \tau)}$, where*

$$D(m, \tau) = (m\tau)^{6m\tau} (2m)^{4m(m+1)\tau}, \quad \Delta(m, \tau) = \frac{1}{2} m(m+1) \left(1 - \left(1 - \frac{1}{m} \right)^\tau \right).$$

For the proof of this version of Vinogradov's mean value theorem, see [9, Ch. VI, §1].

PROOF OF THEOREM 1. Shifting the interval of summation to at most $d \leq N^{1/15}$ terms, we obtain the inequality $|S| \leq |S_1| + d$, where the sum S_1 has the same type as the initial sum S and satisfies the additional condition $c \equiv 0 \pmod{d}$.

Further, suppose that $h = [N^{1/4}] + 1$, $1 \leq x, y \leq h$ and define q_ε as in lemma 1 (the precise value of ε will be chosen later). Then

$$\begin{aligned} S_1 &= \sum'_{c < n + q_\varepsilon xy \leq c + N} e_q(a(n + q_\varepsilon xy)^* + b(n + q_\varepsilon xy)) = \\ &= \sum'_{1 \leq n \leq N} e_q(a(n + c + q_\varepsilon xy)^* + b(n + c + q_\varepsilon xy)) + 2\theta q_\varepsilon xy, \quad |\theta| \leq 1. \end{aligned}$$

Summing over $1 \leq x, y \leq h$ we get

$$|S_1| \leq h^{-2} \sum'_{1 \leq n \leq N} |W| + h^2 q_\varepsilon,$$

where

$$W = W(n) = \sum_{x,y=1}^h e_q(a(n+c+q_\varepsilon xy)^* + bq_\varepsilon xy).$$

Setting $v \equiv (n+c)^* \pmod{q}$ for brevity and using lemma 1, we obtain

$$\begin{aligned} |W| &= \left| \sum_{x,y=1}^h e_q(a_1 xy + a_2 (xy)^2 + \dots + a_m (xy)^m) \right| = \\ &= \left| \sum_{x,y=1}^h e(\alpha_1 xy + \alpha_2 (xy)^2 + \dots + \alpha_m (xy)^m) \right|, \end{aligned}$$

where $e(z) = e^{2\pi iz}$, $m = [2\varepsilon^{-1}]$ and

$$a_1 \equiv q_\varepsilon(b - av^2) \pmod{q}, \quad a_r \equiv (-1)^r av^{r+1} q_\varepsilon^r \pmod{q}, \quad r = 2, 3, \dots, m, \quad \alpha_r = \frac{a_r}{q}.$$

Taking $k = m\tau$ (where the value of τ will be chosen later) and applying Hölder's inequality to W as in [9, Ch. VI, §1], we find that

$$\begin{aligned} |W|^{2k} &\leq h^{2k-1} \sum_{x=1}^h \left| \sum_{y=1}^h e(\alpha_1 xy + \dots + \alpha_m x^m y^m) \right|^{2k} = \\ &= h^{2k-1} \sum_{x=1}^h \sum_{\lambda_1, \dots, \lambda_m} J_{k,m}(h; \lambda_1, \dots, \lambda_m) e(\alpha_1 \lambda_1 x + \dots + \alpha_m \lambda_m x^m) \leq \\ &\leq h^{2k-1} \sum_{\lambda_1, \dots, \lambda_m} J_{k,m}(h; \lambda_1, \dots, \lambda_m) \left| \sum_{x=1}^h e(\alpha_1 \lambda_1 x + \dots + \alpha_m \lambda_m x^m) \right|, \end{aligned}$$

where λ_r runs through some set of values, $|\lambda_r| < \Lambda_r$, $\Lambda_r = kh^r$. Obviously,

$$\sum_{\lambda_1, \dots, \lambda_m} J_{k,m}(h; \lambda_1, \dots, \lambda_m) \leq h^{2k}.$$

Using Hölder's inequality again, we obtain:

$$\begin{aligned} |W|^{4k^2} &\leq h^{2k(2k-1)} \left(\sum_{\lambda_1, \dots, \lambda_m} J_{k,m}(h; \lambda_1, \dots, \lambda_m) \right)^{2k-1} \times \\ &\times \sum_{\lambda_1, \dots, \lambda_m} J_{k,m}(h; \lambda_1, \dots, \lambda_m) \left| \sum_{x=1}^h e(\alpha_1 \lambda_1 x + \dots + \alpha_m \lambda_m x^m) \right|^{2k} \leq \\ &\leq h^{4k(2k-1)} J_{k,m}(h) \sum_{\lambda_1, \dots, \lambda_m} \left| \sum_{x=1}^h e(\alpha_1 \lambda_1 x + \dots + \alpha_m \lambda_m x^m) \right|^{2k}, \end{aligned}$$

where λ_r in the last sum runs through the entire interval $|\lambda_r| < \Lambda_r$ ($r = 1, \dots, m$). Further,

$$\begin{aligned} |W|^{4k^2} &\leq \\ &\leq h^{4k(2k-1)} J_{k,m}(h) \sum_{\lambda_1, \dots, \lambda_m} \sum_{\mu_1, \dots, \mu_m} J_{k,m}(h; \mu_1, \dots, \mu_m) e(\alpha_1 \lambda_1 \mu_1 + \dots + \alpha_m \lambda_m \mu_m) \leq \\ &\leq h^{4k(2k-1)} J_{k,m}(h) \sum_{\mu_1, \dots, \mu_m} J_{k,m}(h; \mu_1, \dots, \mu_m) \left| \sum_{\lambda_1} e(\alpha_1 \mu_1 \lambda_1) \right| \cdots \left| \sum_{\lambda_m} e(\alpha_m \mu_m \lambda_m) \right|. \end{aligned}$$

By lemma 2,

$$|W|^{4k^2} \leq h^{4k(2k-1)} J_{k,m}^2(h) V_1 \cdots V_m,$$

where

$$V_r = \sum_{|\mu_r| < \Lambda_r} \min(2\Lambda_r, \|\alpha_r \mu_r\|^{-1}), \quad r = 1, 2, \dots, m.$$

Obviously, the trivial bound for V_r is $(2\Lambda_r)^2$. At the same time, representing α_r as incontractible fraction of the form

$$\frac{A_r}{Q_r}, \quad Q_r = p_1^{\gamma_1} \cdots p_s^{\gamma_s}, \quad \gamma_\ell = \max(0, \alpha_\ell - r\beta_\ell), \quad (4)$$

we obtain:

$$\begin{aligned} V_r &= 6 \left(\frac{2\Lambda_r}{Q_r} + 1 \right) (2\Lambda_r + Q_r \ln Q_r) < (2\Lambda_r)^2 \delta_r, \\ \delta_r &= 6(\log q) \left(\frac{1}{Q_r} + \frac{1}{2\Lambda_r} \right) \left(1 + \frac{Q_r}{2\Lambda_r} \right) = 6(\log q) \left(\frac{1}{\sqrt{Q_r}} + \frac{\sqrt{Q_r}}{2\Lambda_r} \right)^2. \end{aligned}$$

Hence, setting $\Delta_r = \min(1, \delta_r)$, we get

$$|W|^{4k^2} \leq h^{4k(2k-1)} J_{k,m}^2(h) \Delta \prod_{r=1}^m (2\Lambda_r)^2 = (2k)^{2m} h^{4k(2k-1)+m(m+1)} J_{k,m}^2(h) \Delta,$$

where $\Delta = \Delta_1 \cdots \Delta_m$. Using lemma 4, we obtain

$$\begin{aligned} J_{k,m}^2(h) &\leq (m\tau)^{12m\tau} (2m)^{4m(m+1)\tau} h^{4k-m(m+1)} \left(1 - \left(1 - \frac{1}{m} \right)^\tau \right) \Delta = \\ &= k^{12k} (2m)^{4k(m+1)} h^{4k-m(m+1)} \left(1 - \left(1 - \frac{1}{m} \right)^\tau \right) \Delta, \\ |W|^{4k^2} &\leq C(k, m) h^{8k^2+m(m+1)} \left(1 - \frac{1}{m} \right)^\tau \Delta, \quad C(k, m) = k^{12k} (2m)^{4k(m+1)} (2k)^{2m}. \end{aligned}$$

Now let us choose

$$\varepsilon = c \frac{\log N}{\log q}, \quad c = \frac{1}{7}, \quad r_j = \lfloor c_j \varepsilon^{-1} \rfloor, \quad j = 1, 2, \quad c_1 = \frac{1}{3}, \quad c_2 = \frac{2}{3},$$

and note that $N = q^{7\varepsilon}$, $r_1 \geq 1$, $r_2 - r_1 \geq 1$. Suppose now that $r_1 < r \leq r_2$. Then, by (4), we have $qq_\varepsilon^{-r} \leq Q_r \leq qq_\varepsilon^{-1} < q$ and hence

$$\frac{1}{Q_r} \leq \frac{q_\varepsilon^r}{q} \leq \frac{(dq_\varepsilon)^r}{q} \leq \frac{(dq_\varepsilon)^{r_2}}{q}.$$

Obviously, $q^{\varepsilon r_2} \leq q^{2/3}$, $d^{r_2} \leq N^{r_2/15} = q^{7\varepsilon r_2/15} \leq q^{14/15}$, so we have $Q_r \leq q^{-1/45}$. Similarly,

$$\frac{Q_r}{4\Lambda_r^2} < \frac{q}{h^{2r}} \leq \frac{q}{h^{2(r_1+1)}} \leq \frac{q}{N^{(r_1+1)/2}} \leq q^{-1/6}.$$

Hence,

$$\delta_r \leq 6(\log q)(q^{-1/90} + q^{-1/12})^2 < q^{-1/50},$$

$$\Delta \leq \prod_{r=r_1+1}^{r_2} \delta_r < q^{-(r_2-r_1)/50} \leq q^{-\varepsilon^{-1}/200} = \exp\left(-\frac{7}{200} \frac{(\log q)^2}{\log N}\right),$$

$$|W|^{4k^2} < C(k, m)h^{8k^2+m(m+1)(1-\frac{1}{m})^\tau} \exp\left(-\frac{7}{200} \frac{(\log q)^2}{\log N}\right).$$

Now we set $\tau = \kappa m$, $\kappa = 10$. Then we have

$$m(m+1)\left(1 - \frac{1}{m}\right)^\tau \leq e^{-10}m(m+1) < \frac{1}{91} \frac{(\log q)^2}{\log N},$$

$$h^{m(m+1)(1-\frac{1}{m})^\tau} \leq \exp\left(\frac{1}{360} \frac{(\log q)^2}{\log N}\right),$$

$$|W|^{4k^2} < C(k, m)h^{8k^2} \exp\left(-\frac{1}{31.5} \frac{(\log q)^2}{\log N}\right).$$

Since $0 < \varepsilon < \frac{1}{14}$ then $m \geq 28$ and

$$C(k, m)^{1/(4k^2)} = (10m^2)^{\frac{1}{30m^2}} (20m^2)^{\frac{1}{200m^3}} (2m)^{\frac{m+1}{10m^2}} < 1.02.$$

Thus we have

$$|W| < 1.02h^2 \exp\left(-\frac{1}{126k^2} \frac{(\log N)^3}{(\log q)^2}\right) < 1.02h^2 \exp\left(-\frac{1}{156^4} \frac{(\log N)^3}{(\log q)^2}\right),$$

and, finally,

$$|S_1| < 1.02N \exp\left(-\frac{1}{156^4} \frac{(\log N)^3}{(\log q)^2}\right), \quad |S| \leq |S_1| + d < N \exp\left(-\frac{1}{160^4} \frac{(\log N)^3}{(\log q)^2}\right).$$

Theorem is proved. \square

The restriction $d^{15} \leq N$ can be weakened slightly, but the price is the narrowing of the interval for N and the loss of precision of the estimate. Namely, the following assertion is true.

THEOREM 2. Suppose that $0 < \delta < 0.1$ is any fixed number, $q \geq q_0(\delta)$ and let

$$\gamma_1 = 1200\delta^{-2}(\log(1/\delta))^{2/3}, \quad \max(d^{2+\delta}, e^{\gamma_1(\log q)^{2/3}}) \leq N \leq q^{\delta/20}.$$

Then the inequality (2) holds with $\gamma = 201^{-4}\delta^6(\log(1/\delta))^2$.

PROOF. Setting $m = \lceil 2\varepsilon^{-1} \rceil$, $r_j = \lceil c_j \varepsilon^{-1} \rceil$, $j = 1, 2$,

$$\varepsilon = c \frac{\log N}{\log q} \quad c = \frac{\delta}{5} \left(1 - \frac{\delta}{15}\right), \quad c_1 = \frac{2\delta}{5} \left(1 - \frac{\delta}{20}\right), \quad c_2 = \frac{2\delta}{5}$$

in the above calculations and noting that $q^\varepsilon = N^c$, $d \leq N^{1/(2+\delta)}$, we find

$$q_\varepsilon h^2 \leq 2\sqrt{N}dq^\varepsilon \leq 2N^\xi,$$

where

$$\xi = \frac{1}{2} + \frac{1}{2+\delta} + c < \frac{1}{2} + \frac{1}{2} \left(1 - \frac{\delta}{2} + \frac{\delta^2}{4}\right) + \frac{\delta}{5} = 1 - \frac{\delta}{20} \left(1 - \frac{5\delta}{2}\right) < 1 - \frac{\delta}{40}.$$

Further, if $r_1 < r \leq r_2$ then

$$\begin{aligned} \frac{1}{Q_r} &\leq \frac{(dq^\varepsilon)^{r_2}}{q} = q^\eta, \\ \eta &= -1 + \varepsilon r_2 \left(1 + \frac{1}{c(2+\delta)}\right) \leq -1 + c_2 \left(1 + \frac{1}{c(2+\delta)}\right) = \\ &= \frac{2\delta}{5} + \frac{1 - (1 - \frac{\delta}{15})(1 + \frac{\delta}{2})}{(1 - \frac{\delta}{15})(1 + \frac{\delta}{2})} = -\frac{\delta}{30} \frac{1 - \frac{31\delta}{5} + \frac{2\delta^2}{5}}{(1 - \frac{\delta}{15})(1 + \frac{\delta}{2})} < -\frac{\delta}{150}, \end{aligned}$$

$$\frac{Q_r}{4\Lambda_r^2} < \frac{q}{h^{2r}} \leq \frac{q}{N^{(r_1+1)/2}} = q^\vartheta,$$

$$\vartheta = 1 - \frac{\varepsilon}{2c} (r_1 + 1) \leq 1 - \frac{c_1}{2c} = -\frac{\delta}{60} \frac{1}{1 - \frac{\delta}{15}} < -\frac{\delta}{60},$$

so we have

$$\delta_r < 24(\log q)q^{-\delta/150} < q^{-\delta/160}, \quad \Delta < q^{-\delta(r_2-r_1)/160} < \exp\left(-\frac{\delta^2}{3200} \frac{(\log q)^2}{\log N}\right).$$

Taking $\tau = \kappa m$, $\kappa = \lceil 4\log(1/\delta) \rceil + 14$ we find that

$$h^{m(m+1)(1-\frac{1}{m})^\tau} \leq h^{e^{-\kappa}m(m+1)} < \exp\left(\frac{5e^{-\kappa}}{4c^2} \frac{(\log q)^2}{\log N}\right) < \exp\left(\frac{\delta^2}{6400} \frac{(\log q)^2}{\log N}\right)$$

and hence

$$|W|^{4k^2} < C(k, m)h^{8k^2} \exp\left(-\frac{\delta^2}{6400} \frac{(\log q)^2}{\log N}\right),$$

where $C(k, m)$ is defined above. Now it remains to check the inequalities

$$\frac{\delta^2}{4k^2 \cdot 6400} = \frac{\delta^2 \kappa^{-2} c^4}{640^2} \frac{(\log q)^4}{(\log N)^4} > \frac{\delta^6}{200^4} (\log(1/\delta))^{-2} \frac{(\log q)^4}{(\log N)^4}.$$

Theorem 2 is proved. \square

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